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# EULER-LAGRANGE EQUATIONS FOR COMPOSITION FUNCTIONALS IN CALCULUS OF VARIATIONS ON TIME SCALES

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Abstract. In this paper we consider the problem of the calculus of variations for a functional which is the composition of a certain scalar function $H$ with the delta integral of a vector valued field $f$, i.e., of the form $H\left(\int_a^b f(t, x^\sigma(t), x^\Delta(t))\Delta t\right)$. Euler-Lagrange equations, natural boundary conditions for such problems as well as a necessary optimality condition for isoperimetric problems, on a general time scale, are given. A number of corollaries are obtained, and several examples illustrating the new results are discussed in detail.

1. **Introduction.** The calculus on time scales was introduced by Bernd Aulbach and Stefan Hilger in 1988 [6]. The new theory bridges the divide and extends the traditional areas of continuous and discrete analysis and the various dialects of $q$-calculus [14] into a single theory [11, 12, 20]. The calculus of variations on time scales was born with the works [2, 8, 18] and has interesting applications in Economics [3, 4, 5, 15, 26]. Currently, several researchers are getting interested in the new theory and contributing to its development (see, e.g., [7, 9, 10, 16, 21, 22, 23, 24, 25]). The present work is dedicated to the study of general (non-classical) problems of calculus of variations on an arbitrary time scale $\mathbb{T}$. As a particular case, by choosing $\mathbb{T} = \mathbb{R}$, one gets the generalized calculus of variations [13] with functionals of the form

$$H\left(\int_a^b f(t, x(t), x'(t))dt\right),$$

where $f$ has $n$ components and $H$ has $n$ independent variables. Cases of calculus of variations as these appear in practical applications (see [13] and the references given therein) but cannot be solved using the classical theory. Therefore, an extension of this theory is needed.

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The paper is organized as follows. In Section 2, some preliminaries on time scales are presented. Our results are given in Section 3 and Section 4. We begin Section 3 by formulating the general (non-classical) problem of calculus of variations (1) on an arbitrary time scale. We obtain a general formula for the Euler-Lagrange equations and natural boundary conditions for the general problem (Theorem 3.2), which are then applied to the product (Corollary 3.4) and the quotient (Corollary 3.7). In Section 4 we prove a necessary optimality condition for the general isoperimetric problem (Theorem 4.3 and Theorem 4.5). Throughout the paper several examples illustrating the new results are discussed in detail.

2. **Preliminaries.** The following definitions and theorems will serve as a short introduction to the calculus of time scales; they can be found in [11, 12].

A nonempty closed subset of $\mathbb{R}$ is called a *time scale* and it is denoted by $\mathbb{T}$. The real numbers ($\mathbb{R}$), the integers ($\mathbb{Z}$), the natural numbers ($\mathbb{N}$), the $h$-numbers ($h\mathbb{Z} := \{hz | z \in \mathbb{Z}\}$, where $h > 0$ is a fixed real number), and the $q$-numbers ($q^{\mathbb{N}_0} := \{q^k | k \in \mathbb{N}_0\}$, where $q > 1$ is a fixed real number) are examples of time scales, as are $\{0, \frac{1}{2}, 1\}$, $[2, 3] \cup \mathbb{N}$, and $[-1, 1] \cup [2, 3]$, where $[-1, 1]$ and $[2, 3]$ are real number intervals. We assume that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

**Definition 2.1.** For $t \in \mathbb{T}$ we define he *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \text{ for all } t \in \mathbb{T},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \text{ for all } t \in \mathbb{T}.$$

In this definition we consider $\sigma(M) = M$ if $\mathbb{T}$ has a maximum $M$ and $\rho(m) = m$ if $\mathbb{T}$ has a minimum $m$.

A point $t \in \mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense* and *left-scattered* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ and $\rho(t) < t$, respectively. Points that are simultaneously right-scattered and left-scattered are called *isolated*. Points that are simultaneously right-dense and left-dense are called *dense*.

The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t, \text{ for all } t \in \mathbb{T}.$$

**Example 2.2.** If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = \rho(t) = t$ and $\mu(t) = 0$. If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\rho(t) = t - 1$, and $\mu(t) = 1$. On the other hand, if $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$ is a fixed real number, then we have $\sigma(t) = qt$, $\rho(t) = q^{-1}t$, and $\mu(t) = (q - 1)t$.

**Definition 2.3.** A time scale $\mathbb{T}$ is called *regular* if the following two conditions are satisfied:

(i) $\sigma(\rho(t)) = t$, for all $t \in \mathbb{T}$; and
(ii) $\rho(\sigma((t)) = t$, for all $t \in \mathbb{T}$.

Following [11], let us define

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if} \quad \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases}$$

**Definition 2.4.** We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *delta differentiable* at $t \in \mathbb{T}^\kappa$ if there exists a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there is a neighborhood $U$ of $t$ (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \text{ for all } s \in U.$$

We call $f^{\Delta}(t)$ the *delta derivative* of $f$ at $t$ and $f$ is said *delta differentiable* on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.

**Remark 2.5.** If $t \in \mathbb{T} \setminus \mathbb{T}^{\kappa}$, then $f^{\Delta}(t)$ is not uniquely defined, since for such a point $t$, small neighborhoods $U$ of $t$ consist only of $t$ and, besides, we have $\sigma(t) = t$. For this reason, maximal left-scattered points are omitted in Definition 2.4.

Note that in right-dense points $f^{\Delta}(t) = lim_{s \to t} \frac{f(t)-f(s)}{t-s}$, provided this limit exists, and in right-scattered points $f^{\Delta}(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)}$, provided $f$ is continuous at $t$.

**Example 2.6.** If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f'(t)$, i.e., the delta derivative coincides with the usual one. If $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$. If $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, then $f^{\Delta}(t) = \frac{f(qt)-f(t)}{(q-1)t}$, i.e., we get the usual derivative of quantum calculus [19].

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at right-dense points and if its left-sided limit exists at left-dense points. We denote the set of all rd-continuous functions by $\mathrm{C}_{\mathrm{rd}}$ and the set of all delta differentiable functions with rd-continuous derivative by $\mathrm{C}^1_{\mathrm{rd}}$.

Now we introduce the concept of integral for a function $f : \mathbb{T} \to \mathbb{R}$.

Let $a, b \in \mathbb{T}$ with $a \leq b$. We define the closed interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] := \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Open intervals and half-open intervals in $\mathbb{T}$ are defined accordingly. In what follows all intervals will be time scale intervals.

It is known that rd-continuous function possess an *antiderivative*, i.e., there exists a function $F$ with $F^{\Delta} = f$, and in this case the *delta integral* is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a)$$

for all $a, b \in \mathbb{T}$.

The delta integral has the following properties:

(i) if $f \in C_{rd}$ and $t \in \mathbb{T}^{\kappa}$, then

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t)\,;$$

(ii) if $a, b \in \mathbb{T}$ and $f, g \in C_{rd}$, then

$$\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^{\Delta}(t)g(t)\Delta t\,,$$

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t;$$

(iii) if $[a, b]$ consists of only isolated points, then

$$\int_a^b f(t)\Delta t = \sum_{t \in [a,b)} \mu(t)f(t).$$

**Example 2.7.** Let $a, b \in \mathbb{T}$ with $a < b$. If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, where the integral on the right-hand side is the classical Riemann integral. If $\mathbb{T} = \mathbb{Z}$, then $\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k)$. If $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, then $\int_a^b f(t)\Delta t = (1-q)\sum_{t \in [a,b)} tf(t)$.

The Dubois-Reymond lemma of the calculus of variations on time scales will be useful for our purposes.

**Lemma 2.8.** *(Lemma of Dubois-Reymond [8]) Let $\mathbb{T} = [a, b]$ be a time scale with at least three points and let $g \in C_{rd}$, $g : \mathbb{T}^\kappa \to \mathbb{R}$. Then,*

$$\int_a^b g(t) \cdot \eta^\Delta(t)\Delta t = 0 \quad \text{for all } \eta \in C^1_{rd} \text{ with } \eta(a) = \eta(b) = 0$$

*if and only if $g(t) = c$ on $\mathbb{T}^\kappa$ for some $c \in \mathbb{R}$.*

3. **Euler-Lagrange equations.** Let $\mathbb{T}$ be a time scale. Throughout we let $A, B \in \mathbb{T}$ with $A < B$. For an interval $[c, d] \cap \mathbb{T}$ we simply write $[c, d]$. We also abbreviate $f \circ \sigma$ by $f^\sigma$. Now let $[a, b]$, with $a, b \in \mathbb{T}$ and $b < B$, be a subinterval of $[A, B]$.

The general (non-classical) problem of the calculus of variations on time scales under our consideration consists of minimizing or maximizing a functional of the form

$$\mathcal{L}[x] = H\left(\int_a^b f_1(t, x^\sigma(t), x^\Delta(t))\Delta t, \ldots, \int_a^b f_n(t, x^\sigma(t), x^\Delta(t))\Delta t\right), \qquad (1)$$
$$(x(a) = x_a), \quad (x(b) = x_b)$$

over all $x \in C^1_{rd}$. Using parentheses around the end-point conditions means that these conditions may or may not be present. We assume that:

(i) the function $H : \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives with respect to its arguments and we denote them by $H'_i$, $i = 1, \ldots, n$;
(ii) functions $(t, y, v) \to f_i(t, y, v)$ from $[a, b] \times \mathbb{R}^2$ to $\mathbb{R}$, $i = 1, \ldots, n$, have partial continuous derivatives with respect to $y, v$ for all $t \in [a, b]$ and we denote them by $f_{iy}$, $f_{iv}$;
(iii) $f_i$, $f_{iy}$, $f_{iv}$, $i = 1, \ldots, n$, are rd-continuous in $t$ for all $x \in C^1_{rd}$.

A function $x \in C^1_{rd}$ is said to be an admissible function provided that it satisfies the end-points conditions (if any is given).

Let us consider the following norm in $C^1_{rd}$:

$$\|x\|_1 = \sup_{t\in[a,b]} |x^\sigma(t)| + \sup_{t\in[a,b]} |x^\Delta(t)|.$$

**Definition 3.1.** An admissible function $\tilde{x}$ is said to be a *weak local minimizer* (respectively *weak local maximizer*) for (1) if there exists $\delta > 0$ such that $\mathcal{L}[\tilde{x}] \le \mathcal{L}[x]$ (respectively $\mathcal{L}[\tilde{x}] \ge \mathcal{L}[x]$) for all admissible $x$ with $\|x - \tilde{x}\|_1 < \delta$.

Next theorem gives necessary optimality conditions for problem (1).

**Theorem 3.2.** *If $\tilde{x}$ is a weak local solution of the problem (1), then the Euler-Lagrange equation*

$$\sum_{i=1}^n H'_i(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}])\left(f^\Delta_{iv}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - f_{iy}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t))\right) = 0 \qquad (2)$$

*holds for all $t \in [a, b]^\kappa$, where $\mathcal{F}_i[\tilde{x}] = \int_a^b f_i(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t))\Delta t$, $i = 1, \ldots, n$. Moreover, if $x(a)$ is not specified, then*

$$\sum_{i=1}^n H'_i(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) f_{iv}(a, \tilde{x}^\sigma(a), \tilde{x}^\Delta(a)) = 0; \qquad (3)$$

*and if $x(b)$ is not specified, then*

$$\sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}],\dots,\mathcal{F}_n[\tilde{x}])\Big(f_{iv}(\rho(b),\tilde{x}^\sigma(\rho(b)),\tilde{x}^\Delta(\rho(b))) + \int_{\rho(b)}^{b} f_{iy}(t,\tilde{x}^\sigma(t),\tilde{x}^\Delta(t))\Delta t\Big) = 0. \quad (4)$$

*Proof.* Suppose that $\mathcal{L}[x]$ has a weak local extremum at $\tilde{x}$. For an admissible variation $h \in C^1_{rd}$ we define a function $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(\varepsilon) = \mathcal{L}[(\tilde{x}+\varepsilon h)]$. We do not require $h(a) = 0$ or $h(b) = 0$ in case $x(a)$ or $x(b)$, respectively, is free (it is possible that both are free). A necessary condition for $\tilde{x}$ to be an extremizer for $\mathcal{L}[x]$ is given by $\phi'(\varepsilon)|_{\varepsilon=0} = 0$. Using the chain rule for obtaining the derivative of a composed function we get

$$\phi'(\varepsilon)|_{\varepsilon=0} = \sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}],\dots,\mathcal{F}_n[\tilde{x}]) \int_a^b \left[f_{iy}(\bullet)h^\sigma(t) + f_{iv}(\bullet)h^\Delta(t)\right]\Delta t,$$

where $(\bullet) = \big(t,\tilde{x}^\sigma(t),\tilde{x}^\Delta(t)\big)$. Integration by parts of the first term of the integrand gives

$$\int_a^b f_{iy}(\bullet)h^\sigma(t)\Delta t = \int_a^t f_{iy}(\circ)\Delta\tau h(t)|_{t=a}^{t=b} - \int_a^b \left(\int_a^t f_{iy}(\circ)\Delta\tau h^\Delta(t)\right)\Delta t,$$

where $(\circ) = \big(\tau,\tilde{x}^\sigma(\tau),\tilde{x}^\Delta(\tau)\big)$.The necessary condition $\phi'(\varepsilon)|_{\varepsilon=0} = 0$ can be written as

$$0 = \int_a^b h^\Delta(t)\sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}],\dots,\mathcal{F}_n[\tilde{x}])\left(f_{iv}(\bullet) - \int_a^t f_{iy}(\circ)\Delta\tau\right)\Delta t + \sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}],\dots,\mathcal{F}_n[\tilde{x}])\left(\int_a^t f_{iy}(\circ)\Delta\tau h(t)\right)\Bigg|_{t=a}^{t=b}. \quad (5)$$

In particular, equation (5) holds for all variations which are zero at both ends. For all such $h$'s the second term in (5) is zero and by the Dubois-Reymond Lemma 2.8, we have

$$\sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}],\dots,\mathcal{F}_n[\tilde{x}])\left(f_{iv}(\bullet) - \int_a^t f_{iy}(\circ)\Delta\tau\right)\Delta t = c, \quad (6)$$

for some $c \in \mathbb{R}$ and all $t \in [a,b]$. Hence, equation (2) holds for all $t \in [a,b]^\kappa$. Equation (5) must be satisfied for all admissible values of $h(a)$ and $h(b)$. Consequently, equations (5) and (6) imply that

$$0 = \left(c + \sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}],\dots,\mathcal{F}_n[\tilde{x}])\int_a^b f_{iy}(\bullet)\Delta t\right)h(b) - \left(c + \sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}],\dots,\mathcal{F}_n[\tilde{x}])\int_a^a f_{iy}(\bullet)\Delta t\right)h(a).$$

From the properties of the delta integral and from (6), it follows that

$$0 = h(b) \left\{ \sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) \left( f_{iv}(\rho(b), \tilde{x}^\sigma(\rho(b)), \tilde{x}^\Delta(\rho(b)) \right.\right.$$
$$\left.\left. + \int_{\rho(b)}^{b} f_{iy}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \Delta t \right) \right\} - h(a) \left( \sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) f_{iv}(a, \tilde{x}^\sigma(a), \tilde{x}^\Delta(a)) \right). \tag{7}$$

If $x(t)$ is not preassigned at either end-point, then $h(a)$ and $h(b)$ are both completely arbitrary and we conclude that their coefficients in (7) must each vanish. It follows that condition (3) holds when $x(a)$ is not given, and condition (4) holds when $x(b)$ is not given. □

**Remark 3.3.** Let $\mathbb{T}$ be a regular time scale. Then from the properties of the delta integral we have

$$\int_{\rho(b)}^{b} f_{iy}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \Delta t = \mu(\rho(t)) f_{iy}(\rho(b), \tilde{x}^\sigma(\rho(b)), \tilde{x}^\Delta(\rho(b))).$$

Therefore (4) can be written in the form

$$\sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) \left\{ f_{iv}(\rho(b), \tilde{x}^\sigma(\rho(b)), \tilde{x}^\Delta(\rho(b))) \right.$$
$$\left. + \mu(\rho(t)) f_{iy}(\rho(b), \tilde{x}^\sigma(\rho(b)), \tilde{x}^\Delta(\rho(b))) \right\}.$$

Choosing $\mathbb{T} = \mathbb{R}$ in Theorem 3.2 we immediately obtain Theorem 3.1 and Equation (4.1) in [13]. The Euler-Lagrange Equation for the product functional can be deduced from Theorem 3.2.

**Corollary 3.4.** *If $\tilde{x}$ is a solution of the problem*

$$\mathcal{L}[x] = \left( \int_a^b f_1(t, x^\sigma(t), x^\Delta(t)) \Delta t \right) \left( \int_a^b f_2(t, x^\sigma(t), x^\Delta(t)) \Delta t \right),$$
$$(x(a) = x_a), \quad (x(b) = x_b),$$

*then the Euler-Lagrange equation*

$$\mathcal{F}_2[\tilde{x}] \left( f_{1v}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - f_{1y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \right)$$
$$+ \mathcal{F}_1[\tilde{x}] \left( f_{2v}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - f_{2y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \right) = 0$$

*holds for all $t \in [a, b]^\kappa$. Moreover, if $x(a)$ is not specified, then*

$$\mathcal{F}_2[\tilde{x}] f_{1v}(a, \tilde{x}^\sigma(a), \tilde{x}^\Delta(a)) + \mathcal{F}_1[\tilde{x}] f_{2v}(a, \tilde{x}^\sigma(a), \tilde{x}^\Delta(a)) = 0;$$

*if $x(b)$ is not specified, then*

$$\mathcal{F}_2[\tilde{x}] \left( f_{1v}(\rho(b), \tilde{x}^\sigma(\rho(b)), \tilde{x}^\Delta(\rho(b))) + \int_{\rho(b)}^{b} f_{1y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \Delta t \right)$$
$$+ \mathcal{F}_1[\tilde{x}] \left( f_{2v}(\rho(b), \tilde{x}^\sigma(\rho(b)), \tilde{x}^\Delta(\rho(b))) + \int_{\rho(b)}^{b} f_{2y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \Delta t \right) = 0.$$

**Remark 3.5.** In the particular case $\mathbb{T} = \mathbb{R}$, Corollary 3.4 gives Equation (3.17) in [13].

**Example 3.6.** Consider the problem

$$\begin{gathered}\text{minimize} \quad \mathcal{L}[x] = \left(\int_0^1 (x^\Delta(t))^2 \Delta t\right)\left(\int_0^1 t x^\Delta(t)\Delta t\right) \\ x(0)=0, \quad x(1)=1.\end{gathered} \tag{8}$$

If $\tilde{x}$ is a local minimum of (8), then the Euler-Lagrange equation must hold, i.e.,

$$2\tilde{x}^{\Delta\Delta}(t)Q_2 + Q_1 = 0, \tag{9}$$

where

$$Q_1 = \mathcal{F}_1[\tilde{x}] = \int_0^1 (\tilde{x}^\Delta(t))^2\Delta t, \quad Q_2 = \mathcal{F}_2[\tilde{x}] = \int_0^1 t\tilde{x}^\Delta(t)\Delta t.$$

If $Q_2 = 0$, then also $Q_1 = 0$. This contradicts the fact that on any time scale a global minimizer for the problem

$$\begin{gathered}\text{minimize} \quad \mathcal{F}_1[x] = \int_0^1 (x^\Delta(t))^2\Delta t \\ x(0)=0, \quad x(1)=1\end{gathered}$$

is $\bar{x}(t) = t$ and $\mathcal{F}_1[\bar{x}] = 1$. Hence, $Q_2 \neq 0$ and equation (9) implies that candidate solutions for problem (8) are those satisfying the delta differential equation

$$\tilde{x}^{\Delta\Delta}(t) = -\frac{Q_1}{2Q_2} \tag{10}$$

subject to boundary conditions $x(0) = 0$ and $x(1) = 1$. Solving equation (10) we obtain

$$x(t) = -\frac{Q_1}{2Q_2}\int_0^t \tau\Delta\tau + 1 + \frac{Q_1}{2Q_2}\int_0^1 \tau\Delta\tau.$$

Therefore, a solution of (10) depends on the time scale. Let us consider, for example, $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \left\{0, \frac{1}{2}, 1\right\}$. On $\mathbb{T} = \mathbb{R}$ we obtain

$$x(t) = -\frac{Q_1}{4Q_2}t^2 + \frac{4Q_2 + Q_1}{4Q_2}t. \tag{11}$$

Substituting (11) into functionals $\mathcal{F}_1$ and $\mathcal{F}_2$ gives

$$\begin{cases}\frac{48Q_2^2+Q_1^2}{48Q_2^2} = Q_1 \\ \frac{12Q_2-Q_1}{24Q_2} = Q_2.\end{cases} \tag{12}$$

Solving the system of equations (12) we obtain

$$\begin{cases}Q_1 = 0 \\ Q_2 = 0,\end{cases} \qquad \begin{cases}Q_1 = \frac{4}{3} \\ Q_2 = \frac{1}{3}.\end{cases}$$

Therefore,

$$\tilde{x}(t) = -t^2 + 2t$$

is a candidate extremizer for problem (8) on $\mathbb{T} = \mathbb{R}$. Note that nothing can be concluded as to whether $\tilde{x}$ gives a minimum, a maximum, or neither of these for $\mathcal{L}$. The solution of (10) on $\mathbb{T} = \left\{0, \frac{1}{2}, 1\right\}$ is

$$x(t) = \begin{cases}0 & \text{if } t = 0 \\ \frac{1}{2} + \frac{Q_1}{16Q_2} & \text{if } t = \frac{1}{2} \\ 1 & \text{if } t = 1.\end{cases} \tag{13}$$

Constants $Q_1$ and $Q_2$ are determined by substituting (13) into functionals $\mathcal{F}_1$ and $\mathcal{F}_2$. The resulting system of equations is

$$\begin{cases} 1+\frac{Q_1^2}{64Q_2^2}=Q_1 \\ \frac{1}{4}-\frac{Q_1}{32Q_2}=Q_2. \end{cases} \tag{14}$$

Since system of equations (14) has no real solutions, we conclude that there exists no extremizer for problem (8) on $\mathbb{T}=\left\{0,\frac{1}{2},1\right\}$ among the set of functions that we consider to be admissible.

Assuming that the denominator does not vanish, the Euler-Lagrange equation for the quotient problem can be deduced from Theorem 3.2.

**Corollary 3.7.** *If $\tilde{x}$ is a solution of the problem*

$$\mathcal{L}[x]=\frac{\int_a^b f_1(t,x^\sigma(t),x^\Delta(t))\Delta t}{\int_a^b f_2(t,x^\sigma(t),x^\Delta(t))\Delta t},$$
$$(x(a)=x_a), \quad (x(b)=x_b),$$

*then the Euler-Lagrange equation*

$$\begin{aligned} f_{1v}^\Delta(t,\tilde{x}^\sigma(t),\tilde{x}^\Delta(t)) &- f_{1y}(t,\tilde{x}^\sigma(t),\tilde{x}^\Delta(t)) \\ &- Q\left(f_{2v}^\Delta(t,\tilde{x}^\sigma(t),\tilde{x}^\Delta(t)) - f_{2y}(t,\tilde{x}^\sigma(t),\tilde{x}^\Delta(t))\right)=0 \end{aligned}$$

*holds for all $t\in[a,b]^\kappa$, where $Q=\frac{\mathcal{F}_1[\tilde{x}]}{\mathcal{F}_2[\tilde{x}]}$. Moreover, if $x(a)$ is not specified, then*

$$f_{1v}(a,\tilde{x}^\sigma(a),\tilde{x}^\Delta(a)) - Qf_{2v}(a,\tilde{x}^\sigma(a),\tilde{x}^\Delta(a))=0;$$

*if $x(b)$ is not specified, then*

$$\begin{aligned} f_{1v}(\rho(b),\tilde{x}^\sigma(\rho(b)),\tilde{x}^\Delta(\rho(b))) &+ \int_{\rho(b)}^b f_{1y}(t,\tilde{x}^\sigma(t),\tilde{x}^\Delta(t))\Delta t \\ - Q&\left(f_{2v}(\rho(b),\tilde{x}^\sigma(\rho(b)),\tilde{x}^\Delta(\rho(b))) + \int_{\rho(b)}^b f_{2y}(t,\tilde{x}^\sigma(t),\tilde{x}^\Delta(t))\Delta t\right)=0. \end{aligned}$$

**Remark 3.8.** In the particular situation $\mathbb{T}=\mathbb{R}$, Corollary 3.7 gives Equation (3.21) in [13].

**Example 3.9.** Consider the problem

$$\begin{gathered} \text{minimize} \quad \mathcal{L}[x]=\frac{\int_0^2 (x^\Delta(t))^2\Delta t}{\int_0^2 (x^\Delta(t)+(x^\Delta(t))^2)\Delta t}, \\ x(0)=0, \quad x(2)=4. \end{gathered} \tag{15}$$

If $\tilde{x}$ is a local minimizer for (15), then the Euler-Lagrange equation must hold, i.e.,

$$0=[2\tilde{x}^\Delta(t)-Q(1+2\tilde{x}^\Delta(t))]^\Delta, \quad t\in[0,2]^\kappa,$$

where

$$Q=\frac{\int_0^2 (\tilde{x}^\Delta(t))^2\Delta t}{\int_0^2 (\tilde{x}^\Delta(t)+(\tilde{x}^\Delta(t))^2)\Delta t}.$$

Therefore,

$$0=2\tilde{x}^{\Delta\Delta}(t)-Q2\tilde{x}^{\Delta\Delta}(t), \quad t\in[0,2]^\kappa.$$

Thus $\tilde{x}^{\Delta\Delta}(t) = 0$ or $Q = 1$. The solution of the delta differential equation

$$x^{\Delta\Delta}(t) = 0,$$
$$x(0) = 0, \quad x(2) = 4$$

does not depend on the time scale and it is $\tilde{x}(t) = 2t$. Observe that $\mathcal{L}[\tilde{x}] = \frac{2}{3} < 1$. Therefore, $\tilde{x}$ is a candidate local minimizer for problem (15).

**Example 3.10.** Consider the problem

$$\begin{gathered}\text{extremize} \quad \mathcal{L}[x] = \frac{\int_0^1 t x^{\Delta}(t)\Delta t}{\int_0^1 (x^{\Delta}(t))^2 \Delta t}, \\ x(0) = 0, \quad x(1) = 1.\end{gathered} \tag{16}$$

The Euler-Lagrange equation for this problem is

$$0 = 1 - 2Q x^{\Delta\Delta}(t),$$

where $Q$ is the value of functional $\mathcal{L}$ in a solution of (16). Since $Q \neq 0$, it follows that

$$x^{\Delta\Delta}(t) = \frac{1}{2Q}. \tag{17}$$

Solving equation (17) we obtain

$$x(t) = \frac{1}{2Q}\int_0^t \tau\Delta\tau + 1 - \frac{1}{2Q}\int_0^1 \tau\Delta\tau.$$

Therefore, a solution of (17) depends on the time scale. Let us consider, for example, $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \{0, \frac{1}{2}, 1\}$. On $\mathbb{T} = \mathbb{R}$ we obtain

$$x(t) = \frac{1}{4Q}t^2 + \frac{4Q - 1}{4Q}t. \tag{18}$$

Substituting (18) into functional $\mathcal{L}$ yields

$$\frac{24Q^2 + 2Q}{48Q^2 + 1} = Q. \tag{19}$$

Solving equation (19) we obtain $Q \in \left\{\frac{1}{4} - \frac{\sqrt{3}}{6}, 0, \frac{1}{4} + \frac{\sqrt{3}}{6}\right\}$. Therefore,

$$x_1(t) = \frac{3}{3 - 2\sqrt{3}}t^2 + \frac{2\sqrt{3}}{2\sqrt{3} - 3}t$$

is a candidate local minimizer while

$$x_2(t) = \frac{3}{3 + 2\sqrt{3}}t^2 + \frac{2\sqrt{3}}{2\sqrt{3} + 3}t$$

is a candidate local maximizer for problem (16) on $\mathbb{T} = \mathbb{R}$.
The solution of (17) on $\mathbb{T} = \left\{0, \frac{1}{2}, 1\right\}$ is

$$x(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{1}{2} - \frac{1}{16Q} & \text{if } t = \frac{1}{2} \\ 1 & \text{if } t = 1. \end{cases} \tag{20}$$

The constant $Q$ is determined by substituting (20) into $\mathcal{L}$. The resulting equation is

$$\frac{1}{4} + \frac{1}{32Q} = Q + \frac{1}{64Q}. \tag{21}$$

Solving (21) we obtain $Q \in \left\{\frac{1}{8} - \frac{\sqrt{2}}{8}, \frac{1}{8} + \frac{\sqrt{2}}{8}\right\}$ and stationary functions are

$$x_1(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{\sqrt{2}}{2\sqrt{2}-2} & \text{if } t = \frac{1}{2} \\ 1 & \text{if } t = 1, \end{cases} \tag{22}$$

and

$$x_2(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{\sqrt{2}}{2\sqrt{2}+2} & \text{if } t = \frac{1}{2} \\ 1 & \text{if } t = 1. \end{cases} \tag{23}$$

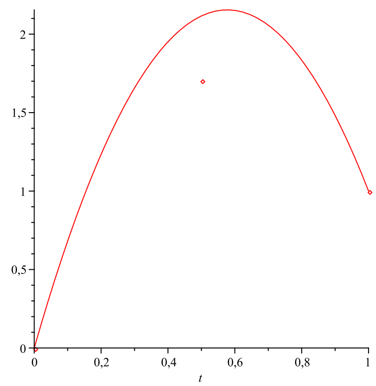


FIGURE 1. The extremal minimizer of Example 3.10 for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \{0, \frac{1}{2}, 1\}$.

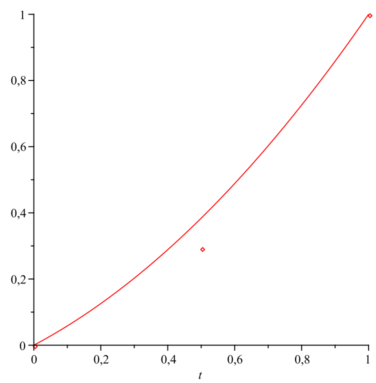


FIGURE 2. The extremal maximizer of Example 3.10 for for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \{0, \frac{1}{2}, 1\}$.

Therefore (22) is a candidate local minimizer while (23) is a candidate local maximizer for problem (16) on $\mathbb{T} = \left\{0, \frac{1}{2}, 1\right\}$.

**Example 3.11.** Consider the problem

$$\text{extremize} \quad \mathcal{L}[x] = \frac{\int_a^b [(x^\Delta(t))^2 - q(t)(x^\sigma(t))^2]\Delta t}{\int_a^b (x^\sigma(t))^2 \Delta t}, \qquad x(a) = 0, \quad x(b) = 0, \tag{24}$$

where $q : [a, b] \to \mathbb{R}$ is a continuous function. The Euler-Lagrange equation for this problem is

$$x^{\Delta\Delta}(t) + q(t)x^\sigma(t) + Qx^\sigma(t) = 0, \tag{25}$$

subject to

$$x(a) = 0, \quad x(b) = 0, \tag{26}$$

where $Q$ is the value of functional $\mathcal{L}$ in a solution of (24). It is easily seen that (25)–(26) is a case of the Sturm-Liouville eigenvalue problem on time scales (see [1] and [17]). It follows that the problem of determining eigenfunctions of (25) subject to (26) is equivalent to the problem of determining functions satisfying (26) which render $\mathcal{L}$ stationary.

4. **Isoperimetric problems.** Let us consider now the general (non-classical) isoperimetric problem on time scales. The problem consists of minimizing or maximizing

$$\mathcal{L}[x] = H\left(\int_a^b f_1(t, x^\sigma(t), x^\Delta(t))\Delta t, \ldots, \int_a^b f_n(t, x^\sigma(t), x^\Delta(t))\Delta t\right), \tag{27}$$

in the class of functions $x \in C^1_{rd}$ satisfying the boundary conditions

$$x(a) = x_a, \quad x(b) = x_b \tag{28}$$

and the constraint

$$\mathcal{K}[x] = P\left(\int_a^b g_1(t, x^\sigma(t), x^\Delta(t))\Delta t, \ldots, \int_a^b g_m(t, x^\sigma(t), x^\Delta(t))\Delta t\right) = k, \tag{29}$$

where $x_a$, $x_b$, $k$ are given real numbers. We assume that:

(i) functions $H : \mathbb{R}^n \to \mathbb{R}$ and $P : \mathbb{R}^m \to \mathbb{R}$ have continuous partial derivatives with respect to their arguments and we denote them by $H_i'$, $i = 1, \ldots, n$, and $P_i'$, $i = 1, \ldots, m$;

(ii) functions $(t, y, v) \to f_i(t, y, v)$, $i = 1, \ldots, n$, and $(t, y, v) \to g_j(t, y, v)$, $j = 1, \ldots, m$, from $[a, b] \times \mathbb{R}^2$ to $\mathbb{R}$ have partial continuous derivatives with respect to $y, v$ for all $t \in [a, b]$ and we denote them by $f_{iy}$, $f_{iv}$ and $g_{jy}$, $g_{jv}$;

(iii) $f_i$, $f_{iy}$, $f_{iv}$, $i = 1, \ldots, n$, and $g_j$, $g_{jy}$, $g_{jv}$, $j = 1, \ldots, m$, are rd-continuous in $t$ for all $x \in C^1_{rd}$.

**Definition 4.1.** An admissible function $\tilde{x}$ is said to be a *weak local minimizer* (respectively *weak local maximizer*) for the isoperimetric problem (27)–(29) if there exists $\delta > 0$ such that $\mathcal{L}[\tilde{x}] \leq \mathcal{L}[x]$ (respectively $\mathcal{L}[\tilde{x}] \geq \mathcal{L}[x]$) for all admissible $x$ satisfying the boundary conditions (28), the isoperimetric constraint (29), and $\|x - \tilde{x}\|_1 < \delta$.

**Definition 4.2.** We say that $\tilde{x}$ is an extremal for $\mathcal{K}$ if

$$\sum_{i=1}^m P_i'(\mathcal{G}_1[\tilde{x}], \ldots, \mathcal{G}_m[\tilde{x}])\left(g_{iv}(\bullet) - \int_a^t g_{iy}(\circ)\Delta\tau\right) = c,$$

where $(\bullet) = \left(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)\right)$ and $(\circ) = \left(\tau, \tilde{x}^\sigma(\tau), \tilde{x}^\Delta(\tau)\right)$, for some constant $c$ and for all $t \in [a, b]$. An extremizer (i.e., a weak local minimizer or a weak local maximizer) for the problem (27)–(29) that is not an extremal for $\mathcal{K}$ is said to be a normal extremizer; otherwise (i.e., if it is an extremal for $\mathcal{K}$), the extremizer is said to be abnormal.

**Theorem 4.3.** *If $\tilde{x}$ is a normal extremizer for the isoperimetric problem (27)–(29), then there exists a real $\lambda$ such that*

$$\begin{aligned}\sum_{i=1}^n & H_i'(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}])\left(f_{iv}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - f_{iy}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t))\right) \\ &- \lambda \sum_{i=1}^m P_i'(\mathcal{G}_1[\tilde{x}], \ldots, \mathcal{G}_m[\tilde{x}])\left(g_{iv}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - g_{iy}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t))\right) = 0 \end{aligned} \tag{30}$$

*for all $t \in [a, b]^\kappa$.*

*Proof.* Consider a variation of $\tilde{x}$, say $\bar{x} = \tilde{x} + \varepsilon_1 h_1 + \varepsilon_2 h_2$, where $h_i \in C^1_{rd}$ and $h_i(a) = h_i(b) = 0$, $i = 1, 2$, and $\varepsilon_i$ is a sufficiently small parameter ($\varepsilon_1$ and $\varepsilon_2$ must

be such that $\|\bar{x}-\tilde{x}\|_1<\delta$ for some $\delta>0$). Here, $h_1$ is an arbitrary fixed function and $h_2$ is a fixed function that will be chosen later. Define the real function

$$\bar{K}(\varepsilon_1,\varepsilon_2)=\mathcal{K}[\bar{x}]=P\left(\int_a^b g_1(t,\bar{x}^\sigma(t),\bar{x}^\Delta(t))\Delta t,\dots,\int_a^b g_m(t,\bar{x}^\sigma(t),\bar{x}^\Delta(t))\Delta t\right)-k.$$

We have

$$\left.\frac{\partial\bar{K}}{\partial\varepsilon_2}\right|_{(0,0)}=\sum_{i=1}^m P_i'(\mathcal{G}_1[\tilde{x}],\dots,\mathcal{G}_m[\tilde{x}])\int_a^b\left[g_{iy}(\bullet)h_2^\sigma(t)+g_{iv}(\bullet)h_2^\Delta(t)\right]\Delta t,$$

where $(\bullet)=\big(t,\tilde{x}^\sigma(t),\tilde{x}^\Delta(t)\big)$. Since $h_2(a)=h_2(b)=0$, integration by parts gives

$$\int_a^b h_2^\Delta(t)\sum_{i=1}^m P_i'(\mathcal{G}_1[\tilde{x}],\dots,\mathcal{G}_m[\tilde{x}])\left(g_{iv}(\bullet)-\int_a^t g_{iy}(\circ)\Delta\tau\right)\Delta t,$$

where $(\circ)=\big(\tau,\tilde{x}^\sigma(\tau),\tilde{x}^\Delta(\tau)\big)$. By Lemma 2.8, there exists $h_2$ such that $\left.\frac{\partial\bar{K}}{\partial\varepsilon_2}\right|_{(0,0)}\neq 0$. Since $\bar{K}(0,0)=0$, by the implicit function theorem we conclude that there exists a function $\varepsilon_2$ defined in the neighborhood of zero, such that $\bar{K}(\varepsilon_1,\varepsilon_2(\varepsilon_1))=0$, i.e., we may choose a subset of variations $\bar{x}$ satisfying the isoperimetric constraint.
Let us now consider the real function

$$\bar{L}(\varepsilon_1,\varepsilon_2)=\mathcal{L}[\bar{x}]=H\left(\int_a^b f_1(t,\bar{x}^\sigma(t),\bar{x}^\Delta(t))\Delta t,\dots,\int_a^b f_n(t,\bar{x}^\sigma(t),\bar{x}^\Delta(t))\Delta t\right).$$

By hypothesis, $(0,0)$ is an extremal of $\bar{L}$ subject to the constraint $\bar{K}=0$ and $\nabla\bar{K}(0,0)\neq\mathbf{0}$. By the Lagrange multiplier rule, there exists some real $\lambda$ such that $\nabla(\bar{L}(0,0)-\lambda\bar{K}(0,0))=\mathbf{0}$. Having in mind that $h_1(a)=h_1(b)=0$, we can write

$$\left.\frac{\partial\bar{L}}{\partial\varepsilon_1}\right|_{(0,0)}=\int_a^b h_1^\Delta(t)\sum_{i=1}^n H_i'(\mathcal{F}_1[\tilde{x}],\dots,\mathcal{F}_n[\tilde{x}])\left(f_{iv}(\bullet)-\int_a^t f_{iy}(\circ)\Delta\tau\right)\Delta t$$

and

$$\left.\frac{\partial\bar{K}}{\partial\varepsilon_1}\right|_{(0,0)}=\int_a^b h_1^\Delta(t)\sum_{i=1}^m P_i'(\mathcal{G}_1[\tilde{x}],\dots,\mathcal{G}_m[\tilde{x}])\left(g_{iv}(\bullet)-\int_a^t g_{iy}(\circ)\Delta\tau\right)\Delta t.$$

Therefore,

$$\begin{aligned}\int_a^b h_1^\Delta(t)\Bigg\{&\sum_{i=1}^n H_i'(\mathcal{F}_1[\tilde{x}],\dots,\mathcal{F}_n[\tilde{x}])\left(f_{iv}(\bullet)-\int_a^t f_{iy}(\circ)\Delta\tau\right)\\&-\lambda\sum_{i=1}^m P_i'(\mathcal{G}_1[\tilde{x}],\dots,\mathcal{G}_m[\tilde{x}])\left(g_{iv}(\bullet)-\int_a^t g_{iy}(\circ)\Delta\tau\right)\Bigg\}\Delta t=0.\end{aligned}\tag{31}$$

As (31) holds for any $h_1$, by Lemma 2.8, we have

$$\begin{aligned}&\sum_{i=1}^n H_i'(\mathcal{F}_1[\tilde{x}],\dots,\mathcal{F}_n[\tilde{x}])\left(f_{iv}(\bullet)-\int_a^t f_{iy}(\circ)\Delta\tau\right)\\&\qquad-\lambda\sum_{i=1}^m P_i'(\mathcal{G}_1[\tilde{x}],\dots,\mathcal{G}_m[\tilde{x}])\left(g_{iv}(\bullet)-\int_a^t g_{iy}(\circ)\Delta\tau\right)=c,\end{aligned}\tag{32}$$

for some $c\in\mathbb{R}$. Applying the delta derivative to both sides of equation (32), we get (30). $\square$

**Remark 4.4.** Choosing $H, P : \mathbb{R} \to \mathbb{R}$ and $H = P = id$ in Theorem 4.3 we immediately obtain Theorem 3.4 in [17] and a particular case of Theorem 3.4 in [21].

One can easily cover abnormal extremizers within our result by introducing an extra multiplier $\lambda_0$.

**Theorem 4.5.** *If $\tilde{x}$ is an extremizer for the isoperimetric problem (27)–(29), then there exist two constants $\lambda_0$ and $\lambda$, not both zero, such that*

$$\lambda_0 \sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) \left(f_{iv}^{\Delta}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - f_{iy}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t))\right)$$
$$- \lambda \sum_{i=1}^{m} P_i'(\mathcal{G}_1[\tilde{x}], \ldots, \mathcal{G}_m[\tilde{x}]) \left(g_{iv}^{\Delta}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - g_{iy}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t))\right) = 0 \quad (33)$$

*for all $t \in [a, b]^\kappa$.*

*Proof.* Following the proof of Theorem 4.3, since $(0, 0)$ is an extremal of $\bar{L}$ subject to the constraint $\bar{K} = 0$, the extended Lagrange multiplier rule (see, for instance, [27, Theorem 4.1.3]) asserts the existence of reals $\lambda_0$ and $\lambda$, not both zero, such that $\nabla(\lambda_0 \bar{L}(0, 0) - \lambda \bar{K}(0, 0)) = \mathbf{0}$. Therefore,

$$\lambda_0 \left.\frac{\partial \bar{L}}{\partial \varepsilon_1}\right|_{(0,0)} - \lambda \left.\frac{\partial \bar{K}}{\partial \varepsilon_1}\right|_{(0,0)} = 0$$

$$\Leftrightarrow \int_a^b h_1^\Delta(t) \left\{ \lambda_0 \sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) \left( f_{iv}(\bullet) - \int_a^t f_{iy}(\circ) \Delta\tau \right) \right.$$
$$\left. - \lambda \sum_{i=1}^{m} P_i'(\mathcal{G}_1[\tilde{x}], \ldots, \mathcal{G}_m[\tilde{x}]) \left( g_{iv}(\bullet) - \int_a^t g_{iy}(\circ) \Delta\tau \right) \right\} \Delta t = 0. \quad (34)$$

Since (34) holds for any $h_1$, it follows by Lemma 2.8 that

$$\lambda_0 \sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) \left( f_{iv}(\bullet) - \int_a^t f_{iy}(\circ) \Delta\tau \right)$$
$$- \lambda \sum_{i=1}^{m} P_i'(\mathcal{G}_1[\tilde{x}], \ldots, \mathcal{G}_m[\tilde{x}]) \left( g_{iv}(\bullet) - \int_a^t g_{iy}(\circ) \Delta\tau \right) = c, \quad (35)$$

for some $c \in \mathbb{R}$. The desired condition (33) follows by delta differentiation of (35). □

**Remark 4.6.** If $\tilde{x}$ is a normal extremizer for the isoperimetric problem (27)–(29), then we can choose $\lambda_0 = 1$ in Theorem 4.5 and obtain Theorem 4.3. For abnormal extremizers, Theorem 4.5 holds with $\lambda_0 = 0$. The condition $(\lambda_0, \lambda) \neq \mathbf{0}$ guarantees that Theorem 4.5 is a useful necessary condition.

**Corollary 4.7.** *(i) If $\tilde{x}$ is an extremizer for the isoperimetric problem*

$$\text{extremize} \quad \mathcal{L}[x] = \left( \int_a^b f_1(t, x^\sigma(t), x^\Delta(t)) \Delta t \right) \left( \int_a^b f_2(t, x^\sigma(t), x^\Delta(t)) \Delta t \right),$$
$$x(a) = x_a, \quad x(b) = x_b,$$

*subject to the constraint*

$$\mathcal{K}[x] = \left(\int_a^b g_1(t, x^\sigma(t), x^\Delta(t))\Delta t\right)\left(\int_a^b g_2(t, x^\sigma(t), x^\Delta(t))\Delta t\right) = k,$$

*then there exist two constants $\lambda_0$ and $\lambda$, not both zero, such that*

$$\begin{gathered}\lambda_0 \left\{\mathcal{F}_2[\tilde{x}]\left(f_{1v}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - f_{1y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)\right)\right.\\ \left.+\mathcal{F}_1[\tilde{x}]\left(f_{2v}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - f_{2y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t))\right)\right\}\\ -\lambda\left\{\mathcal{G}_2[\tilde{x}]\left(g_{1v}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - g_{1y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)\right)\right.\\ \left.+\mathcal{G}_1[\tilde{x}]\left(g_{2v}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - g_{2y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t))\right)\right\} = 0\end{gathered}$$

*for all $t \in [a, b]^\kappa$.*

(ii) *Assume that denominators of $\mathcal{L}$ and $\mathcal{K}$ do not vanish. If $\tilde{x}$ is an extremizer for the isoperimetric problem*

$$\textit{extremize} \quad \mathcal{L}[x] = \frac{\int_a^b f_1(t, x^\sigma(t), x^\Delta(t))\Delta t}{\int_a^b f_2(t, x^\sigma(t), x^\Delta(t))\Delta t}, \quad x(a) = x_a, \quad x(b) = x_b,$$

*subject to the constraint*

$$\mathcal{K}[x] = \frac{\int_a^b g_1(t, x^\sigma(t), x^\Delta(t))\Delta t}{\int_a^b g_2(t, x^\sigma(t), x^\Delta(t))\Delta t} = k,$$

*then there exist two constants $\lambda_0$ and $\lambda$, not both zero, such that*

$$\begin{gathered}\lambda_0 \left\{\mathcal{G}_2[\tilde{x}]\left(f_{1v}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - f_{1y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)\right)\right.\\ \left.-\mathcal{G}_2[\tilde{x}]Q_L\left(f_{2v}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - f_{2y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t))\right)\right\}\\ -\lambda\left\{\mathcal{F}_2[\tilde{x}]\left(g_{1v}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - g_{1y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)\right)\right.\\ \left.-\mathcal{F}_2[\tilde{x}]Q_K\left(g_{2v}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - g_{2y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t))\right)\right\} = 0\end{gathered}$$

*holds for all $t \in [a, b]^\kappa$, where $Q_L = \frac{\mathcal{F}_1[\tilde{x}]}{\mathcal{F}_2[\tilde{x}]}$ and $Q_K = \frac{\mathcal{G}_1[\tilde{x}]}{\mathcal{G}_2[\tilde{x}]}$.*

**Example 4.8.** Consider the problem

$$\begin{aligned}\text{extremize} \quad \mathcal{L}[x] = \frac{\int_0^1 (x^\Delta(t))^2\Delta t}{\int_0^1 t x^\Delta(t)\Delta t},\\ x(0) = 0, \quad x(1) = 1,\end{aligned} \tag{36}$$

subject to the constraint

$$\mathcal{K}[x] = \int_0^1 t x^\Delta(t)\Delta t = 1. \tag{37}$$

Since

$$g_v(t, x^\sigma(t), x^\Delta(t)) - \int_0^t g_y(\tau, x_\sigma(\tau), x^\Delta(\tau))\Delta\tau = t$$

there are no abnormal extremals for the problem (36)–(37). Applying Theorem 4.3, we get the delta differential equation

$$2x^{\Delta\Delta} - Q - \lambda = 0, \tag{38}$$

where $Q$ is the value of functional $\mathcal{L}$ in a solution of (36)–(37). Solving equation (38) we obtain

$$x(t) = \frac{Q+\lambda}{2}\int_0^t \tau\Delta\tau + 1 - \frac{Q+\lambda}{2}\int_0^1 \tau\Delta\tau.$$

Therefore, a solution of (38) depends on the time scale. Let us consider, for example, $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\{0,\frac{1}{2},1\}$. On $\mathbb{T}=\mathbb{R}$ we obtain

$$x(t)=3t^2-2t$$

as a candidate local minimizer while on $\mathbb{T}=\{0,\frac{1}{2},1\}$

$$x(t)=\begin{cases}0 & \text{if } t=0\\ -1 & \text{if } t=\frac{1}{2}\\ 1 & \text{if } t=1.\end{cases}$$

is a candidate local minimizer for the problem (36)–(37).

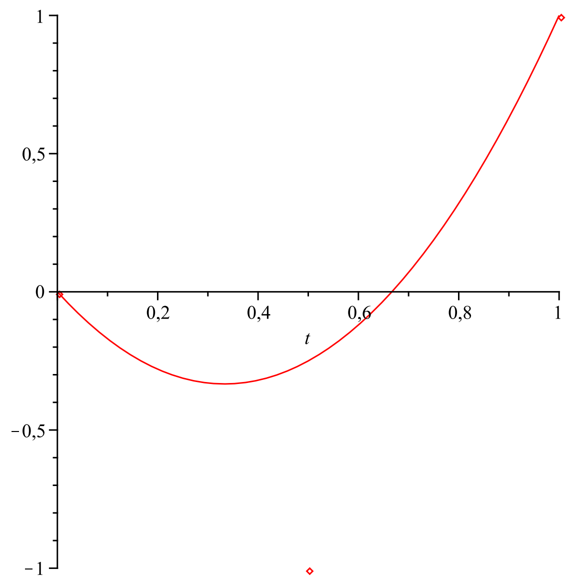


FIGURE 3. The extremal minimizer of Example 4.8 for $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\{0,\frac{1}{2},1\}$.

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